ON LINEAR VOLTERRA DIFFERENCE EQUATIONS WITH INFINITE DELAY

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ABSTRACT. Linear neutral, and especially non-neutral, Volterra difference equations with infinite delay are considered and some new results on the behavior of solutions are established. The results are obtained by the use of appropriate positive roots of the corresponding characteristic equation.

1. PRELIMINARY NOTES

Motivated by the old but significant papers by Driver [3] and Driver, Sasser and Slater [5], a number of relevant papers has recently appeared in the literature. See Frasson and Verduyn Lunel [10], Graef and Qian [11], Kordonis, Niyianni and Philos [16], Kordonis and Philos [18], Kordonis, Philos and Purnaras [21], Philos [27], and Philos and Purnaras [28, 29, 33, 35, 36]. The results in [10, 11, 16, 27, 28, 29, 33, 35] concern the large time behavior of the solutions of several classes of linear autonomous or periodic delay or neutral delay differential equations, while those in [18, 21, 36] are dealing with the behavior of solutions of some linear (neutral or non-neutral) integrodifferential equations with unbounded delay. Note that the method used in [10] is based on resolvent computations and Dunford calculus, while the technique applied in the rest of the papers mentioned above is very simple and is essentially based on elementary calculus. We also notice that the article [10] is very interesting as well as comprehensive.

Along with the work mentioned above for the continuous case, analogous investigations have recently been made for the behavior of the solutions of some classes of linear autonomous or periodic delay or neutral delay difference equations, for the behavior of the solutions of certain linear delay difference equations with continuous variable as well as for the behavior of solutions of a linear Volterra difference equation with infinite delay. See Kordonis and Philos [19], Kordonis, Philos and Purnaras [20], and Philos and Purnaras [30, 31, 32, 34]. For some related results we refer to the papers by De Bruijn [2], Driver, Ladas and Vlahos [4], Györi [12], Norris [25], and Pituk [37, 38].

In [21], Kordonis, Philos and Purnaras obtained some results on the behavior of solutions of linear neutral integrodifferential equations with unbounded delay; the results in [21] extend and improve previous ones given by Kordonis and Philos [18] for the special case of (non-neutral) integrodifferential equations with unbounded delay. In [36], Philos and Purnaras continued the study in [18, 21] and established some further results on the behavior of solutions of linear neutral integrodifferential

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equations with unbounded delay, and, especially, of linear (non-neutral) integrodifferential equations with unbounded delay.

Our purpose in this paper is to give the discrete analogues of the results in [18, 21, 36]. Here, we study the behavior of solutions of linear neutral Volterra difference equations with infinite delay, and, especially, of linear (non-neutral) Volterra difference equations with infinite delay. Our results will be derived by the use of appropriate positive roots of the corresponding characteristic equation. Some of the results of the present paper extend and improve the main results of the authors' previous paper [31].

Neutral, and especially non-neutral, Volterra difference equations with infinite delay have been widely used as mathematical models in mathematical ecology, particularly in population dynamics. Although the bibliography on Volterra integrodifferential equations is quite extended, however there has not yet been analogously much work on the Volterra difference equations. We choose to refer here to the papers by Jaroš and Stavroulakis [13], Kiventidis [15], Kordonis and Philos [17], Ladas, Philos and Sficas [22], and Philos [26] for some results concerning the existence and/or the nonexistence of positive solutions of certain linear Volterra difference equations. Also, for some results on the stability of Volterra difference equations, we typically refer to the papers by Elaydi [6, 7], and Elaydi and Murakami [9] (see, also, the book [8, pp. 239–250]).

For the general background of difference equations, one can refer to the books by Agarwal [1], Elaydi [8], Kelley and Peterson [14], Lakshmikantham and Trigiante [23], Mickens [24], and Sharkovsky, Maistrenko and Romanenko [39].

The paper is organized as follows. Section 2 contains an introduction and some notations. Section 3 is devoted to the statement of the main results (and to some comments on them). The proofs of the main results will be given in Section 4.

2. INTRODUCTION AND NOTATIONS

Throughout the paper, $\mathbf{N}$ stands for the set of all nonnegative integers and $\mathbf{Z}$ stands for the set of all integers. Also, the set of all nonpositive integers will be denoted by $\mathbf{Z}^-$. Moreover, the forward difference operator Δ will be considered to be defined as usual, i.e.

$$\Delta s_n = s_{n+1} - s_n, \ \ n \in \mathbf{N}$$

for any sequence $(s_n)_{n\in\mathbf{N}}$ of real numbers.

Consider the linear neutral Volterra difference equation with infinite delay

$$\text{(E)} \qquad \Delta\left(x_n + \sum_{j=-\infty}^{n-1} G_{n-j}x_j\right) = ax_n + \sum_{j=-\infty}^{n-1} K_{n-j}x_j$$

and, especially, the linear (*non-neutral*) Volterra difference equation with infinite delay

$$\text{(E}_0\text{)} \qquad \Delta x_n = ax_n + \sum_{j=-\infty}^{n-1} K_{n-j}x_j,$$

where *a is a real number, and* $(G_n)_{n\in\mathbf{N}-\{0\}}$ *and* $(K_n)_{n\in\mathbf{N}-\{0\}}$ *are sequences of real numbers. It will be supposed that* $(K_n)_{n\in\mathbf{N}-\{0\}}$ *is not eventually identically* *zero*. Note that (E_0) is a special case of (E), i.e. the special case where the kernel $(G_n)_{n\in\mathbf{N}-\{0\}}$ is identically zero.

Equation (E) can equivalently be written as follows

$$\Delta\left(x_n+\sum_{j=1}^{\infty}G_jx_{n-j}\right)=ax_n+\sum_{j=1}^{\infty}K_jx_{n-j}$$

and, especially, (E_0) can equivalently be written as

$$\Delta x_n=ax_n+\sum_{j=1}^{\infty}K_jx_{n-j}.$$

By a *solution* of the neutral Volterra difference equation (E) (respectively, of the (non-neutral) Volterra difference equation (E_0)), we mean a sequence $(x_n)_{n\in\mathbf{Z}}$ of real numbers which satisfies (E) (resp., (E_0)) for all $n\in\mathbf{N}$.

In the sequel, by S we will denote the (nonempty) set of all sequences $\phi=(\phi_n)_{n\in\mathbf{Z}^-}$ of real numbers such that, for each $n\in\mathbf{N}$,

$$\Phi_n^G\equiv\sum_{j=-\infty}^{-1}G_{n-j}\phi_j=\sum_{j=n+1}^{\infty}G_j\phi_{n-j}\quad\text{and}\quad\Phi_n^K\equiv\sum_{j=-\infty}^{-1}K_{n-j}\phi_j=\sum_{j=n+1}^{\infty}K_j\phi_{n-j}$$

exist in $\mathbf{R}$. In the special case of (E_0), the set S consists of all sequences $\phi=(\phi_n)_{n\in\mathbf{Z}^-}$ of real numbers such that, for each $n\in\mathbf{N}$, Φ_n^K exists in $\mathbf{R}$.

It is clear that, for any given *initial sequence* $\phi=(\phi_n)_{n\in\mathbf{Z}^-}$ in S, there exists a unique solution $(x_n)_{n\in\mathbf{Z}}$ of the difference equation (E) (resp., of (E_0)) which satisfies the *initial condition*

(C) $$x_n=\phi_n\quad\text{for } n\in\mathbf{Z}^-;$$

this solution $(x_n)_{n\in\mathbf{Z}}$ is said to be the solution of the *initial problem* (E)$-$(C) (resp., of the *initial problem* (E_0)$-$(C)) or, more briefly, the solution of (E)$-$(C) (resp., of (E_0)$-$(C)).

With the neutral Volterra difference equation (E) we associate its *characteristic equation*

(*) $$(\lambda-1)\left(1+\sum_{j=1}^{\infty}\lambda^{-j}G_j\right)=a+\sum_{j=1}^{\infty}\lambda^{-j}K_j,$$

which is obtained by seeking solutions of (E) of the form $x_n=\lambda^n$ for $n\in\mathbf{Z}$, where λ is a positive real number. In particular, the *characteristic equation* of the (non-neutral) Volterra difference equation (E_0) is

$(*)_0$ $$\lambda-1=a+\sum_{j=1}^{\infty}\lambda^{-j}K_j.$$

The use of a positive root λ_0 of the characteristic equation (*) with the property

($P(\lambda_0)$) $$\sum_{j=1}^{\infty}\lambda_0^{-j}\left(1+\left|1-\frac{1}{\lambda_0}\right|j\right)|G_j|+\frac{1}{\lambda_0}\sum_{j=1}^{\infty}\lambda_0^{-j}j\,|K_j|<1$$

plays a crucial role in obtaining the results of this paper. In the special case of the (non-neutral) Volterra difference equation (E_0), the property ($P(\lambda_0)$) (of a positive root λ_0 of the characteristic equation $(*)_0$) takes the form

$$(P_0(\lambda_0)) \qquad \frac{1}{\lambda_0}\sum_{j=1}^{\infty}\lambda_0^{-j}j\,|K_j| < 1.$$

In what follows, if λ_0 is a positive root of $(*)$ (resp., of $(*)_0$) with the property ($P(\lambda_0)$) (resp., with the property ($P_0(\lambda_0)$)), we shall denote by $S(\lambda_0)$ the (nonempty) subset of S consisting of all sequences $\phi = (\phi_n)_{n\in\mathbf{Z}^-}$ in S such that $\left(\lambda_0^{-n}\phi_n\right)_{n\in\mathbf{Z}^-}$ is a bounded sequence.

Now, we introduce certain notations which will be used throughout the paper without any further mention. We also give some facts concerning these notations that we shall keep in mind in what follows.

Let λ_0 be a positive root of the characteristic equation $(*)$ with the property ($P(\lambda_0)$). We define

$$\gamma(\lambda_0) = \sum_{j=1}^{\infty}\lambda_0^{-j}\left[1-\left(1-\frac{1}{\lambda_0}\right)j\right]G_j + \frac{1}{\lambda_0}\sum_{j=1}^{\infty}\lambda_0^{-j}jK_j$$

and

$$\mu(\lambda_0) = \sum_{j=1}^{\infty}\lambda_0^{-j}\left(1+\left|1-\frac{1}{\lambda_0}\right|j\right)|G_j| + \frac{1}{\lambda_0}\sum_{j=1}^{\infty}\lambda_0^{-j}j\,|K_j|.$$

Property ($P(\lambda_0)$) together with the hypothesis that $(K_n)_{n\in\mathbf{N}-\{0\}}$ is not eventually identically zero guarantee that

$$0 < \mu(\lambda_0) < 1.$$

Also, because of $|\gamma(\lambda_0)| \le \mu(\lambda_0)$, we have $-1 < \gamma(\lambda_0) < 1$, i.e.

$$0 < 1 + \gamma(\lambda_0) < 2.$$

In the particular case where $(G_n)_{n\in\mathbf{N}-\{0\}}$ and $(K_n)_{n\in\mathbf{N}-\{0\}}$ are nonpositive and λ_0 is less than or equal to 1, because of the fact that $(K_n)_{n\in\mathbf{N}-\{0\}}$ is not eventually identically zero, the property ($P(\lambda_0)$) can be written as $-1 < \gamma(\lambda_0) < 0$, i.e.

$$0 < 1 + \gamma(\lambda_0) < 1.$$

Furthermore, we set

$$\Theta(\lambda_0) = \frac{[1+\mu(\lambda_0)]^2}{1+\gamma(\lambda_0)} + \mu(\lambda_0).$$

We can easily see that $\Theta(\lambda_0)$ is a real number with

$$\Theta(\lambda_0) > 1.$$

Let us consider the special case of the (non-neutral) Volterra difference equation (E_0) and let λ_0 be a positive root of the characteristic equation $(*)_0$ with the property ($P_0(\lambda_0)$). In this case, we define

$$\gamma_0(\lambda_0) = \frac{1}{\lambda_0}\sum_{j=1}^{\infty}\lambda_0^{-j}jK_j$$

and

$$\mu_0(\lambda_0) = \frac{1}{\lambda_0}\sum_{j=1}^{\infty} \lambda_0^{-j} j\, |K_j|.$$

From the property $(\mathrm{P}_0(\lambda_0))$ and the hypothesis that $(K_n)_{n\in\mathbf{N}-\{0\}}$ is not eventually identically zero it follows that

$$0 < \mu_0(\lambda_0) < 1.$$

So, since $|\gamma_0(\lambda_0)| \leq \mu_0(\lambda_0)$, we have $-1 < \gamma_0(\lambda_0) < 1$, namely

$$0 < 1 + \gamma_0(\lambda_0) < 2.$$

If $(K_n)_{n\in\mathbf{N}-\{0\}}$ is assumed to be nonpositive, then, by the fact that $(K_n)_{n\in\mathbf{N}-\{0\}}$ is not eventually identically zero, the property $(\mathrm{P}_0(\lambda_0))$ is equivalent to $-1 < \gamma_0(\lambda_0) < 0$, i.e.

$$0 < 1 + \gamma_0(\lambda_0) < 1.$$

Furthermore, we put

$$\Theta_0(\lambda_0) = \frac{[1+\mu_0(\lambda_0)]^2}{1+\gamma_0(\lambda_0)} + \mu_0(\lambda_0)$$

and we see that $\Theta_0(\lambda_0)$ is a real number with

$$\Theta_0(\lambda_0) > 1.$$

We notice that, in the special case of (E_0), the constants $\gamma(\lambda_0)$, $\mu(\lambda_0)$ and $\Theta(\lambda_0)$, which are defined in the general case of (E), are equal to $\gamma_0(\lambda_0)$, $\mu_0(\lambda_0)$ and $\Theta_0(\lambda_0)$, respectively.

Next, consider again a positive root λ_0 of the characteristic equation $(*)$ with the property $(\mathrm{P}(\lambda_0))$, and let $\phi = (\phi_n)_{n\in\mathbf{Z}^-}$ be an initial sequence in $S(\lambda_0)$. We define

$$\begin{aligned} L(\lambda_0;\phi) &= \phi_0 + \sum_{j=1}^{\infty} G_j \left[\phi_{-j} - \left(1-\frac{1}{\lambda_0}\right)\lambda_0^{-j}\left(\sum_{r=-j}^{-1}\lambda_0^{-r}\phi_r\right)\right] \\ &\quad + \frac{1}{\lambda_0}\sum_{j=1}^{\infty}\lambda_0^{-j}K_j\left(\sum_{r=-j}^{-1}\lambda_0^{-r}\phi_r\right) \end{aligned}$$

and

$$M(\lambda_0;\phi) = \sup_{n\in\mathbf{Z}^-}\left|\lambda_0^{-n}\phi_n - \frac{L(\lambda_0;\phi)}{1+\gamma(\lambda_0)}\right|.$$

From the property $(\mathrm{P}(\lambda_0))$ and the definition of $S(\lambda_0)$ it follows that $L(\lambda_0;\phi)$ is a real number. Moreover, by the definition of $S(\lambda_0)$, $M(\lambda_0;\phi)$ is a nonnegative constant.

Let us concentrate on the special case of the equation (E_0) and consider a positive root λ_0 of the characteristic equation $(*)_0$ with the property $(\mathrm{P}_0(\lambda_0))$ and an initial sequence $\phi = (\phi_n)_{n\in\mathbf{Z}^-}$ in $S(\lambda_0)$. In this special case, we have the constants

$$L_0(\lambda_0;\phi) = \phi_0 + \frac{1}{\lambda_0}\sum_{j=1}^{\infty}\lambda_0^{-j}K_j\left(\sum_{r=-j}^{-1}\lambda_0^{-r}\phi_r\right)$$

and

$$M_0(\lambda_0;\phi) = \sup_{n\in\mathbf{Z}^-} \left| \lambda_0^{-n}\phi_n - \frac{L_0(\lambda_0;\phi)}{1+\gamma_0(\lambda_0)} \right|$$

instead of the constants $L(\lambda_0;\phi)$ and $M(\lambda_0;\phi)$ considered in the general case of the equation (E). Property $(\mathrm{P}_0(\lambda_0))$ and the definition of $S(\lambda_0)$ guarantee that $L_0(\lambda_0;\phi)$ is a real number, and the definition of $S(\lambda_0)$ ensures that $M_0(\lambda_0;\phi)$ is a nonnegative constant.

Another notation used in the paper is the following one

$$N(\lambda_0;\phi) = \sup_{n\in\mathbf{Z}^-} \left(\lambda_0^{-n}\,|\phi_n|\right)$$

for each positive root λ_0 of the characteristic equation $(*)$ (resp., $(*)_0$) with the property $(\mathrm{P}(\lambda_0))$ (resp., $(\mathrm{P}_0(\lambda_0))$) and for any initial sequence $\phi = (\phi_n)_{n\in\mathbf{Z}^-}$ in $S(\lambda_0)$. Clearly, $N(\lambda_0;\phi)$ is a nonnegative constant.

Furthermore, let λ_0 be a positive root of the characteristic equation $(*)$ with the property $(\mathrm{P}(\lambda_0))$ and λ_1 be a positive root of $(*)$ with $\lambda_1 < \lambda_0$. Let also $\phi = (\phi_n)_{n\in\mathbf{Z}^-}$ be an initial sequence in $S(\lambda_0)$. We set

$$U(\lambda_0,\lambda_1;\phi) = \inf_{n\in\mathbf{Z}^-} \left\{ \lambda_1^{-n} \left[\phi_n - \frac{L(\lambda_0;\phi)}{1+\gamma(\lambda_0)}\lambda_0^n \right] \right\}$$

and

$$V(\lambda_0,\lambda_1;\phi) = \sup_{n\in\mathbf{Z}^-} \left\{ \lambda_1^{-n} \left[\phi_n - \frac{L(\lambda_0;\phi)}{1+\gamma(\lambda_0)}\lambda_0^n \right] \right\}.$$

From the definition of $S(\lambda_0)$ and the hypothesis that $\lambda_1 < \lambda_0$ it follows that $U(\lambda_0,\lambda_1;\phi)$ and $V(\lambda_0,\lambda_1;\phi)$ are real constants.

In particular, consider the special case of (E_0). Let λ_0 be a positive root of the characteristic equation $(*)_0$ with the property $(\mathrm{P}_0(\lambda_0))$ and λ_1 be a positive root of $(*)_0$ with $\lambda_1 < \lambda_0$ as well as let $\phi = (\phi_n)_{n\in\mathbf{Z}^-}$ be an initial sequence in $S(\lambda_0)$. In this special case, we consider the real constants

$$U_0(\lambda_0,\lambda_1;\phi) = \inf_{n\in\mathbf{Z}^-} \left\{ \lambda_1^{-n} \left[\phi_n - \frac{L_0(\lambda_0;\phi)}{1+\gamma_0(\lambda_0)}\lambda_0^n \right] \right\}$$

and

$$V_0(\lambda_0,\lambda_1;\phi) = \sup_{n\in\mathbf{Z}^-} \left\{ \lambda_1^{-n} \left[\phi_n - \frac{L_0(\lambda_0;\phi)}{1+\gamma_0(\lambda_0)}\lambda_0^n \right] \right\}$$

in place of $U(\lambda_0,\lambda_1;\phi)$ and $V(\lambda_0,\lambda_1;\phi)$ considered in the general case of (E).

Before closing this section, we will give two well-known definitions. The *trivial solution* of (E) (resp., of (E_0)) is said to be *stable* (*at* 0) if, for each $\epsilon > 0$, there exists $\delta \equiv \delta(\epsilon) > 0$ such that, for any $\phi = (\phi_n)_{n\in\mathbf{Z}^-}$ in S with $\|\phi\| \equiv \sup_{n\in\mathbf{Z}^-} |\phi_n| < \delta$, the solution $(x_n)_{n\in\mathbf{Z}}$ of (E)$-$(C) (resp., of $(\mathrm{E}_0)-$(C)) satisfies $|x_n| < \epsilon$ for all $n \in \mathbf{Z}$. Also, the trivial solution of (E) (resp., of (E_0)) is called *asymptotically stable* (*at* 0) if it is stable (at 0) in the above sense and, in addition, there exists $\delta_0 > 0$ such that, for any $\phi = (\phi_n)_{n\in\mathbf{Z}^-}$ in S with $\|\phi\| < \delta_0$, the solution $(x_n)_{n\in\mathbf{Z}}$ of (E)$-$(C) (resp., of $(\mathrm{E}_0)-$(C)) satisfies $\lim_{n\to\infty} x_n = 0$. Moreover, the trivial solution of (E) (resp., of (E_0)) is called *exponentially stable* (*at* 0) if there exist positive constants Λ and $\eta < 1$ such that, for any $\phi = (\phi_n)_{n\in\mathbf{Z}^-}$ in S with $\|\phi\| < \infty$, the solution

$(x_n)_{n\in\mathbf{Z}}$ of (E)−(C) (resp., of (E$_0$)−(C)) satisfies $|x_n| \leq \Lambda\eta^n \|\phi\|$ for all $n \in \mathbf{N}$ (see Elaydi and Murakami [9]).

3. STATEMENT OF THE MAIN RESULTS

Our first main result is Theorem 1 below, which establishes a useful inequality for solutions of the neutral Volterra difference equation (E). The application of Theorem 1 to the special case of the (non-neutral) Volterra difference equation (E$_0$) leads to Theorem 2 below.

Theorem 1. *Let λ_0 be a positive root of the characteristic equation* $(*)$ *with the property* $(\mathrm{P}(\lambda_0))$. *Then, for any $\phi = (\phi_n)_{n\in\mathbf{Z}^-}$ in $S(\lambda_0)$, the solution $(x_n)_{n\in\mathbf{Z}}$ of* (E)−(C) *satisfies*

$$\left|\lambda_0^{-n}x_n - \frac{L(\lambda_0;\phi)}{1+\gamma(\lambda_0)}\right| \leq \mu(\lambda_0)M(\lambda_0;\phi) \quad \textit{for all } n \in \mathbf{N}.$$

Theorem 2. *Let λ_0 be a positive root of the characteristic equation* $(*)_0$ *with the property* $(\mathrm{P}_0(\lambda_0))$. *Then, for any $\phi = (\phi_n)_{n\in\mathbf{Z}^-}$ in $S(\lambda_0)$, the solution $(x_n)_{n\in\mathbf{Z}}$ of* (E$_0$)−(C) *satisfies*

$$\left|\lambda_0^{-n}x_n - \frac{L_0(\lambda_0;\phi)}{1+\gamma_0(\lambda_0)}\right| \leq \mu_0(\lambda_0)M_0(\lambda_0;\phi) \quad \textit{for all } n \in \mathbf{N}.$$

Theorem 3 below provides an estimate of solutions of the neutral Volterra difference equation (E) that leads to a stability criterion for the *trivial solution* of (E). By applying Theorem 3 to the special case of the (non-neutral) Volterra difference equation (E$_0$), one can be led to the subsequent theorem, i.e. Theorem 4.

Theorem 3. *Let λ_0 be a positive root of the characteristic equation* $(*)$ *with the property* $(\mathrm{P}(\lambda_0))$. *Then, for any $\phi = (\phi_n)_{n\in\mathbf{Z}^-}$ in $S(\lambda_0)$, the solution $(x_n)_{n\in\mathbf{Z}}$ of* (E)−(C) *satisfies*

$$|x_n| \leq \Theta(\lambda_0)N(\lambda_0;\phi)\lambda_0^n \quad \textit{for all } n \in \mathbf{N}.$$

Moreover, the trivial solution of (E) *is stable (at 0) if $\lambda_0 = 1$ and it is asymptotically stable (at 0) if $\lambda_0 < 1$. In addition, the trivial solution of* (E) *is exponentially stable (at 0) if $\lambda_0 < 1$.*

Theorem 4. *Let λ_0 be a positive root of the characteristic equation* $(*)_0$ *with the property* $(\mathrm{P}_0(\lambda_0))$. *Then, for any $\phi = (\phi_n)_{n\in\mathbf{Z}^-}$ in $S(\lambda_0)$, the solution $(x_n)_{n\in\mathbf{Z}}$ of* (E$_0$)−(C) *satisfies*

$$|x_n| \leq \Theta_0(\lambda_0)N(\lambda_0;\phi)\lambda_0^n \quad \textit{for all } n \in \mathbf{N}.$$

Moreover, the trivial solution of (E$_0$) *is stable (at 0) if $\lambda_0 = 1$ and it is asymptotically stable (at 0) if $\lambda_0 < 1$. In addition, the trivial solution of* (E$_0$) *is exponentially stable (at 0) if $\lambda_0 < 1$.*

It must be noted that Theorems 2 and 4 for the (non-neutral) Volterra difference equation (E$_0$) can be considered as substiantally improved versions of the main results of the previous authors' paper [31]. One can easily see the connection between Theorems 2 and 4, and the main results in [31].

The following lemma, i.e. Lemma 1, gives sufficient conditions for the characteristic equation $(*)$ to have a (unique) root λ_0 with the property $(\mathrm{P}(\lambda_0))$. The specialization of Lemma 1 to the special case of the characteristic equation $(*)_0$ is formulated below as Lemma 2. We notice that Lemma 2 has been previously proved in the authors' paper [31].

Lemma 1. *Assume that there exists a positive real number* γ *such that*

$$(\mathrm{H}_1) \qquad \sum_{j=1}^{\infty} \gamma^{-j} |G_j| < \infty \quad \text{and} \quad \sum_{j=1}^{\infty} \gamma^{-j} |K_j| < \infty,$$

$$(\mathrm{H}_2) \qquad (1-\gamma)\sum_{j=1}^{\infty} \gamma^{-j} G_j + \sum_{j=1}^{\infty} \gamma^{-j} K_j > \gamma - 1 - a$$

and

$$(\mathrm{H}_3) \qquad \sum_{j=1}^{\infty} \gamma^{-j} \left[1 + \left(1 + \frac{1}{\gamma}\right) j\right] |G_j| + \frac{1}{\gamma}\sum_{j=1}^{\infty} \gamma^{-j} j\, |K_j| \leq 1.$$

Then, in the interval (γ, ∞), *the characteristic equation* $(*)$ *admits a unique root* λ_0; *this root has the property* $(\mathrm{P}(\lambda_0))$.

Lemma 2. *Assume that there exists a positive real number* γ *such that*

$$(\mathrm{H}_1)_0 \qquad \sum_{j=1}^{\infty} \gamma^{-j} |K_j| < \infty,$$

$$(\mathrm{H}_2)_0 \qquad \sum_{j=1}^{\infty} \gamma^{-j} K_j > \gamma - 1 - a$$

and

$$(\mathrm{H}_3)_0 \qquad \frac{1}{\gamma}\sum_{j=1}^{\infty} \gamma^{-j} j\, |K_j| \leq 1.$$

Then, in the interval (γ, ∞), *the characteristic equation* $(*)_0$ *admits a unique root* λ_0; *this root has the property* $(\mathrm{P}_0(\lambda_0))$.

Theorem 5 and Corollary 1 below concern the behavior of solutions of the neutral Volterra difference equation (E), while Theorem 6 and Corollary 2 below are dealing with the behavior of solutions of the (non-neutral) Volterra difference equation (E_0).

Theorem 5. *Suppose that* $(G_n)_{n\in\mathbf{N}-\{0\}}$ *and* $(K_n)_{n\in\mathbf{N}-\{0\}}$ *are nonpositive. Let* λ_0 *be a positive root of the characteristic equation* $(*)$ *with* $\lambda_0 \leq 1$ *and with the property* $(\mathrm{P}(\lambda_0))$. *Let also* λ_1 *be a positive root of* $(*)$ *with* $\lambda_1 < \lambda_0$. *Then, for any* $\phi = (\phi_n)_{n\in\mathbf{Z}^-}$ *in* $S(\lambda_0)$, *the solution* $(x_n)_{n\in\mathbf{Z}}$ *of* (E)–(C) *satisfies*

$$U(\lambda_0, \lambda_1; \phi) \leq \lambda_1^{-n}\left[x_n - \frac{L(\lambda_0;\phi)}{1+\gamma(\lambda_0)}\lambda_0^n\right] \leq V(\lambda_0, \lambda_1; \phi) \quad \text{for all } n \in \mathbf{N}.$$

We immediately observe that the double inequality in the conclusion of Theorem 5 can equivalently be written as follows

$$U(\lambda_0, \lambda_1; \phi) \left(\frac{\lambda_1}{\lambda_0}\right)^n \leq \lambda_0^{-n} x_n - \frac{L(\lambda_0; \phi)}{1+\gamma(\lambda_0)} \leq V(\lambda_0, \lambda_1; \phi) \left(\frac{\lambda_1}{\lambda_0}\right)^n \quad \text{for } n \in \mathbf{N}.$$

Consequently, since $\lambda_1 < \lambda_0$, we obtain

$$\lim_{n\to\infty} \left(\lambda_0^{-n} x_n\right) = \frac{L(\lambda_0; \phi)}{1+\gamma(\lambda_0)},$$

which establishes the following corollary.

Corollary 1. *Suppose that $(G_n)_{n\in\mathbf{N}-\{0\}}$ and $(K_n)_{n\in\mathbf{N}-\{0\}}$ are nonpositive. Let λ_0 be a positive root of the characteristic equation $(*)$ with $\lambda_0 \leq 1$ and with the property $(\mathrm{P}(\lambda_0))$. Assume that $(*)$ has another positive root less than λ_0. Then, for any $\phi = (\phi_n)_{n\in\mathbf{Z}^-}$ in $S(\lambda_0)$, the solution $(x_n)_{n\in\mathbf{Z}}$ of (E)−(C) satisfies*

$$\lim_{n\to\infty} \left(\lambda_0^{-n} x_n\right) = \frac{L(\lambda_0; \phi)}{1+\gamma(\lambda_0)}.$$

Theorem 6. *Suppose that $(K_n)_{n\in\mathbf{N}-\{0\}}$ is nonpositive. Let λ_0 be a positive root of the characteristic equation $(*)_0$ with the property $(\mathrm{P}_0(\lambda_0))$. Let also λ_1 be a positive root of $(*)_0$ with $\lambda_1 < \lambda_0$. Then, for any $\phi = (\phi_n)_{n\in\mathbf{Z}^-}$ in $S(\lambda_0)$, the solution $(x_n)_{n\in\mathbf{Z}}$ of (E_0)−(C) satisfies*

$$U_0(\lambda_0, \lambda_1; \phi) \leq \lambda_1^{-n} \left[x_n - \frac{L_0(\lambda_0; \phi)}{1+\gamma_0(\lambda_0)} \lambda_0^n\right] \leq V_0(\lambda_0, \lambda_1; \phi) \quad \textit{for all } n \in \mathbf{N}.$$

We see that the double inequality in the conclusion of Theorem 6 is equivalently written as

$$U_0(\lambda_0, \lambda_1; \phi) \left(\frac{\lambda_1}{\lambda_0}\right)^n \leq \lambda_0^{-n} x_n - \frac{L_0(\lambda_0; \phi)}{1+\gamma_0(\lambda_0)} \leq V_0(\lambda_0, \lambda_1; \phi) \left(\frac{\lambda_1}{\lambda_0}\right)^n \quad \text{for } n \in \mathbf{N}.$$

So, as $\lambda_1 < \lambda_0$, we have

$$\lim_{n\to\infty} \left(\lambda_0^{-n} x_n\right) = \frac{L_0(\lambda_0; \phi)}{1+\gamma_0(\lambda_0)}.$$

This proves the following corollary.

Corollary 2. *Suppose that $(K_n)_{n\in\mathbf{N}-\{0\}}$ is nonpositive. Let λ_0 be a positive root of the characteristic equation $(*)_0$ with the property $(\mathrm{P}_0(\lambda_0))$. Assume that $(*)_0$ has another positive root less than λ_0. Then, for any $\phi = (\phi_n)_{n\in\mathbf{Z}^-}$ in $S(\lambda_0)$, the solution $(x_n)_{n\in\mathbf{Z}}$ of (E_0)−(C) satisfies*

$$\lim_{n\to\infty} \left(\lambda_0^{-n} x_n\right) = \frac{L_0(\lambda_0; \phi)}{1+\gamma_0(\lambda_0)}.$$

Now, we state two propositions (Propositions 1 and 2) as well as two lemmas (Lemmas 3 and 4). Proposition 1 and Lemma 3 give some useful information about the positive roots of the characteristic equation $(*)$, while Proposition 2 and Lemma

4 are concerned with the special case of the positive roots of the characteristic equation $(*)_0$.

Proposition 1. *Suppose that $(G_n)_{n\in\mathbf{N}-\{0\}}$ and $(K_n)_{n\in\mathbf{N}-\{0\}}$ are nonpositive. Let λ_0 be a positive root of the characteristic equation $(*)$ with $\lambda_0 \le 1$. If there exists another positive root λ_1 of $(*)$ with $\lambda_1 < \lambda_0$ such that*

$$(\mathrm{Q}(\lambda_1)) \qquad \sum_{j=1}^{\infty} \lambda_1^{-j} j\,|G_j| < \infty \quad \textit{and} \quad \sum_{j=1}^{\infty} \lambda_1^{-j} j\,|K_j| < \infty,$$

then λ_0 has the property $(\mathrm{P}(\lambda_0))$.

Proposition 2. *Suppose that $(K_n)_{n\in\mathbf{N}-\{0\}}$ is nonpositive. Let λ_0 be a positive root of the characteristic equation $(*)_0$. If there exists another positive root λ_1 of $(*)_0$ with $\lambda_1 < \lambda_0$ such that*

$$(\mathrm{Q}_0(\lambda_1)) \qquad \sum_{j=1}^{\infty} \lambda_1^{-j} j\,|K_j| < \infty,$$

then λ_0 has the property $(\mathrm{P}_0(\lambda_0))$.

Lemma 3. *Suppose that $(G_n)_{n\in\mathbf{N}-\{0\}}$ and $(K_n)_{n\in\mathbf{N}-\{0\}}$ are nonpositive.*
(I) *If $a = 0$, then $\lambda = 1$ is not a root of the characteristic equation $(*)$.*
(II) *Assume that $a = 0$ and that*

$$(\mathrm{H}_4) \qquad \sum_{j=1}^{\infty} |G_j| \le 1.$$

Then, in the interval $(1, \infty)$, the characteristic equation $(*)$ has no roots.
(III) *Assume that*

$$(\mathrm{H}_5) \qquad \sum_{j=1}^{\infty} j\,|G_j| < \infty,$$

$$(\mathrm{H}_6) \qquad \sum_{j=1}^{\infty} |G_j| + \sum_{j=1}^{\infty} j\,|K_j| \le 1$$

and

$$(\mathrm{H}_7) \qquad \sum_{j=1}^{\infty} |K_j| \ge a.$$

Then, in the interval $(1, \infty)$, the characteristic equation $(*)$ has no roots.
(IV) *Assume that (H_7) holds, and let there exist a positive real number γ with $\gamma < 1$ and $\gamma < a + 1$ so that*

$$(\mathrm{H}_8) \qquad \sum_{j=1}^{\infty} \gamma^{-j} j\,|G_j| < \infty \quad \textit{and} \quad \sum_{j=1}^{\infty} \gamma^{-j} j\,|K_j| < \infty$$

and

$$(\mathrm{H}_9) \qquad (1-\gamma)\sum_{j=1}^{\infty} \gamma^{-j}\,|G_j| + \sum_{j=1}^{\infty} \gamma^{-j}\,|K_j| > a + 1 - \gamma.$$

Moreover, assume that there exists a real number δ *with* $\delta > 0$ *and* $a < \delta < a+1-\gamma$ *such that*

$$(\mathrm{H}_{10}) \qquad (\delta - a)\sum_{j=1}^{\infty}(a+1-\delta)^{-j}\,|G_j| + \sum_{j=1}^{\infty}(a+1-\delta)^{-j}\,|K_j| < \delta.$$

Then: (i) $\lambda = a+1-\delta$ *is not a root of the characteristic equation* $(*)$. (ii) $\lambda = \gamma$ *is not a root of* $(*)$. (iii) *In the interval* $(a+1-\delta, 1]$, $(*)$ *has a unique root.* (iv) *In the interval* $(\gamma, a+1-\delta)$, $(*)$ *has a unique root.* (*Note: We have* $\delta > 0$ *and* $\gamma < a+1-\delta < 1$.)

Lemma 4. *Suppose that* $(K_n)_{n\in\mathbf{N}-\{0\}}$ *is nonpositive.*

(I) $a > -1$ *is a necessary condition for the characteristic equation* $(*)_0$ *to have at least one positive root.*

(II) *The characteristic equation* $(*)_0$ *has no positive roots greater than or equal to* $a+1$.

(III) *Let* $a > -1$ *and let there exist a positive real number* γ *with* $\gamma < a+1$ *so that*

$$(\mathrm{H}_8)_0 \qquad \sum_{j=1}^{\infty}\gamma^{-j} j\,|K_j| < \infty$$

and

$$(\mathrm{H}_9)_0 \qquad \sum_{j=1}^{\infty}\gamma^{-j}\,|K_j| > a+1-\gamma.$$

Moreover, assume that there exists a real number δ *with* $0 < \delta < a+1-\gamma$ *such that*

$$(\mathrm{H}_{10})_0 \qquad \sum_{j=1}^{\infty}(a+1-\delta)^{-j}\,|K_j| < \delta.$$

Then: (i) $\lambda = a+1-\delta$ *is not a root of the characteristic equation* $(*)_0$. (ii) $\lambda = \gamma$ *is not a root of* $(*)_0$. (iii) *In the interval* $(a+1-\delta, a+1)$, $(*)_0$ *has a unique root.* (iv) *In the interval* $(\gamma, a+1-\delta)$, $(*)_0$ *has a unique root.* (*Note: We have* $\gamma < a+1-\delta < a+1$.)

It is an open problem to examine if Theorem 5, Corollary 1 and Proposition 1 remain valid without the restriction that the root λ_0 of the characteristic equation $(*)$ satisfies $\lambda_0 \le 1$. Such a restriction is not a necessity in the non-neutral case (i.e., in Theorem 6, Corollary 2 and Proposition 2).

Our main results can be extended to the more general case of the linear neutral Volterra-delay difference equation with infinite delay

$$\Delta\left(x_n + \sum_{i=1}^{\infty} c_i x_{n-\sigma_i} + \sum_{j=-\infty}^{n-1} G_{n-j}x_j\right) = a x_n + \sum_{i=1}^{\infty} b_i x_{n-\tau_i} + \sum_{j=-\infty}^{n-1} K_{n-j}x_j$$

and, especially, of the linear neutral Volterra-delay difference equation with infinite delay

$$\Delta x_n = ax_n + \sum_{i=1}^{\infty} b_i x_{n-\tau_i} + \sum_{j=-\infty}^{n-1} K_{n-j}x_j,$$

where c_i and b_i $(i = 1, 2, ...)$ are real numbers, and σ_i and τ_i $(i = 1, 2, ...)$ are positive integers with $\sigma_{i_1} \neq \sigma_{i_2}$ and $\tau_{i_1} \neq \tau_{i_2}$ $(i_1, i_2 = 1, 2, ...; i_1 \neq i_2)$.

The neutral Volterra difference equation with infinite delay (E) can be considered as the discrete version of the neutral Volterra integrodifferential equation with unbounded delay

$$(\widehat{\mathrm{E}}) \qquad \left[x(t) + \int_{-\infty}^{t} G(t-s)x(s)ds\right]' = ax(t) + \int_{-\infty}^{t} K(t-s)x(s)ds,$$

where a is a real number, G and K are continuous real-valued functions on the interval $[0, \infty)$, and K is assumed to be not eventually identically zero. In particular, the (non-neutral) Volterra difference equation with infinite delay (E_0) can be viewed as the discrete version of the (non-neutral) Volterra integrodifferential equation with unbounded delay

$$(\widehat{\mathrm{E}}_0) \qquad x'(t) = ax(t) + \int_{-\infty}^{t} K(t-s)x(s)ds.$$

The results obtained in this paper should be looked upon as the discrete analogues of the ones given by Kordonis and Philos [18], Kordonis, Philos and Purnaras [21], and Philos and Purnaras [36], for the neutral Volterra integrodifferential equation with unbounded delay $(\widehat{\mathrm{E}})$ and, especially, for the (non-neutral) Volterra integrodifferential equation with unbounded delay $(\widehat{\mathrm{E}}_0)$.

4. PROOFS OF THE MAIN RESULTS

Proof of Theorem 1. Let $\phi = (\phi_n)_{n\in\mathbf{Z}^-}$ be an initial sequence in $S(\lambda_0)$, and $(x_n)_{n\in\mathbf{Z}}$ be the solution of (E)−(C).

Define

$$y_n = \lambda_0^{-n} x_n \quad \text{for } n \in \mathbf{Z}.$$

Then, for each $n \in \mathbf{N}$, we obtain

$$\begin{aligned} & \Delta\left(x_n + \sum_{j=-\infty}^{n-1} G_{n-j}x_j\right) - ax_n - \sum_{j=-\infty}^{n-1} K_{n-j}x_j \\ \equiv\ & \Delta\left(x_n + \sum_{j=1}^{\infty} G_j x_{n-j}\right) - ax_n - \sum_{j=1}^{\infty} K_j x_{n-j} \\ =\ & \Delta\left[\lambda_0^n\left(y_n + \sum_{j=1}^{\infty} \lambda_0^{-j} G_j y_{n-j}\right)\right] - a\lambda_0^n y_n - \lambda_0^n \sum_{j=1}^{\infty} \lambda_0^{-j} K_j y_{n-j} \end{aligned}$$

$$
\begin{aligned}
&= \lambda_0^n \left[\lambda_0 \Delta \left(y_n + \sum_{j=1}^{\infty} \lambda_0^{-j} G_j y_{n-j} \right) + (\lambda_0 - 1) \left(y_n + \sum_{j=1}^{\infty} \lambda_0^{-j} G_j y_{n-j} \right) \right. \\
&\qquad \left. - a y_n - \sum_{j=1}^{\infty} \lambda_0^{-j} K_j y_{n-j} \right] \\
&= \lambda_0^n \left[\lambda_0 \Delta \left(y_n + \sum_{j=1}^{\infty} \lambda_0^{-j} G_j y_{n-j} \right) + (\lambda_0 - 1 - a) y_n \right. \\
&\qquad \left. + (\lambda_0 - 1) \sum_{j=1}^{\infty} \lambda_0^{-j} G_j y_{n-j} - \sum_{j=1}^{\infty} \lambda_0^{-j} K_j y_{n-j} \right] \\
&= \lambda_0^n \left[\lambda_0 \Delta \left(y_n + \sum_{j=1}^{\infty} \lambda_0^{-j} G_j y_{n-j} \right) - (\lambda_0 - 1) \left(\sum_{j=1}^{\infty} \lambda_0^{-j} G_j \right) y_n \right. \\
&\qquad \left. + \left(\sum_{j=1}^{\infty} \lambda_0^{-j} K_j \right) y_n + (\lambda_0 - 1) \sum_{j=1}^{\infty} \lambda_0^{-j} G_j y_{n-j} - \sum_{j=1}^{\infty} \lambda_0^{-j} K_j y_{n-j} \right] \\
&= \lambda_0^n \left[\lambda_0 \Delta \left(y_n + \sum_{j=1}^{\infty} \lambda_0^{-j} G_j y_{n-j} \right) - (\lambda_0 - 1) \sum_{j=1}^{\infty} \lambda_0^{-j} G_j \left(y_n - y_{n-j} \right) \right. \\
&\qquad \left. + \sum_{j=1}^{\infty} \lambda_0^{-j} K_j \left(y_n - y_{n-j} \right) \right].
\end{aligned}
$$

So, $(x_n)_{n \in \mathbf{Z}}$ satisfies (E) for $n \in \mathbf{N}$ if and only if $(y_n)_{n \in \mathbf{Z}}$ satisfies

$$
\begin{aligned}
\Delta \left(y_n + \sum_{j=1}^{\infty} \lambda_0^{-j} G_j y_{n-j} \right) &= \left(1 - \frac{1}{\lambda_0} \right) \sum_{j=1}^{\infty} \lambda_0^{-j} G_j \left(y_n - y_{n-j} \right) \\
&\quad - \frac{1}{\lambda_0} \sum_{j=1}^{\infty} \lambda_0^{-j} K_j \left(y_n - y_{n-j} \right) \quad \text{for } n \in \mathbf{N}.
\end{aligned} \tag{4.1}
$$

Moreover, the initial condition (C) can equivalently be written as

$$
y_n = \lambda_0^{-n} \phi_n \quad \text{for } n \in \mathbf{Z}^-. \tag{4.2}
$$

Furthermore, we see that (4.1) becomes

$$
\begin{aligned}
\Delta \left(y_n + \sum_{j=1}^{\infty} \lambda_0^{-j} G_j y_{n-j} \right) &= \left(1 - \frac{1}{\lambda_0} \right) \sum_{j=1}^{\infty} \lambda_0^{-j} G_j \Delta \left(\sum_{r=n-j}^{n-1} y_r \right) \\
&\quad - \frac{1}{\lambda_0} \sum_{j=1}^{\infty} \lambda_0^{-j} K_j \Delta \left(\sum_{r=n-j}^{n-1} y_r \right)
\end{aligned}
$$

$$= \Delta\left[\left(1-\frac{1}{\lambda_0}\right)\sum_{j=1}^{\infty}\lambda_0^{-j}G_j\left(\sum_{r=n-j}^{n-1}y_r\right) - \frac{1}{\lambda_0}\sum_{j=1}^{\infty}\lambda_0^{-j}K_j\left(\sum_{r=n-j}^{n-1}y_r\right)\right]$$

for $n \in \mathbf{N}$. Thus, we have

$$y_n + \sum_{j=1}^{\infty}\lambda_0^{-j}G_jy_{n-j} = \left(1-\frac{1}{\lambda_0}\right)\sum_{j=1}^{\infty}\lambda_0^{-j}G_j\left(\sum_{r=n-j}^{n-1}y_r\right) - \frac{1}{\lambda_0}\sum_{j=1}^{\infty}\lambda_0^{-j}K_j\left(\sum_{r=n-j}^{n-1}y_r\right) + \Lambda$$

for every $n \in \mathbf{N}$, where

$$\Lambda = \left(y_0 + \sum_{j=1}^{\infty}\lambda_0^{-j}G_jy_{-j}\right) - \left(1-\frac{1}{\lambda_0}\right)\sum_{j=1}^{\infty}\lambda_0^{-j}G_j\left(\sum_{r=-j}^{-1}y_r\right) + \frac{1}{\lambda_0}\sum_{j=1}^{\infty}\lambda_0^{-j}K_j\left(\sum_{r=-j}^{-1}y_r\right).$$

But, by using (4.2) and taking into account the definition of $L(\lambda_0;\phi)$, we can immediately verify that $\Lambda = L(\lambda_0;\phi)$. Hence, (4.1) takes the following equivalent form

$$y_n + \sum_{j=1}^{\infty}\lambda_0^{-j}G_jy_{n-j} = \left(1-\frac{1}{\lambda_0}\right)\sum_{j=1}^{\infty}\lambda_0^{-j}G_j\left(\sum_{r=n-j}^{n-1}y_r\right) - \frac{1}{\lambda_0}\sum_{j=1}^{\infty}\lambda_0^{-j}K_j\left(\sum_{r=n-j}^{n-1}y_r\right) + L(\lambda_0;\phi) \quad \text{for } n \in \mathbf{N}. \tag{4.3}$$

Next, we set

$$z_n = y_n - \frac{L(\lambda_0;\phi)}{1+\gamma(\lambda_0)} \quad \text{for } n \in \mathbf{Z}.$$

Then, we take into account the definition of $\gamma(\lambda_0)$ to show that (4.3) may equivalently be written as follows

$$z_n + \sum_{j=1}^{\infty}\lambda_0^{-j}G_jz_{n-j} = \left(1-\frac{1}{\lambda_0}\right)\sum_{j=1}^{\infty}\lambda_0^{-j}G_j\left(\sum_{r=n-j}^{n-1}z_r\right) - \frac{1}{\lambda_0}\sum_{j=1}^{\infty}\lambda_0^{-j}K_j\left(\sum_{r=n-j}^{n-1}z_r\right) \quad \text{for } n \in \mathbf{N}. \tag{4.4}$$

On the other hand, the initial condition (4.2) becomes

$$z_n = \lambda_0^{-n}\phi_n - \frac{L(\lambda_0;\phi)}{1+\gamma(\lambda_0)} \quad \text{for } n \in \mathbf{Z}^-. \tag{4.5}$$

Now, by taking into account the definitions of $(y_n)_{n\in\mathbf{Z}}$ and $(z_n)_{n\in\mathbf{Z}}$, we conclude that what we have to prove is that $(z_n)_{n\in\mathbf{Z}}$ satisfies

$$|z_n| \leq \mu(\lambda_0)M(\lambda_0;\phi) \quad \text{for all } n \in \mathbf{N}. \tag{4.6}$$

In the rest of the proof we will establish (4.6). From (4.5) and the definition of $M(\lambda_0;\phi)$ it follows that

$$|z_n| \leq M(\lambda_0;\phi) \quad \text{for } n \in \mathbf{Z}^-. \tag{4.7}$$

We will show that

$$|z_n| \leq M(\lambda_0;\phi) \quad \text{for all } n \in \mathbf{Z}. \tag{4.8}$$

For this purpose, let us consider an arbitrary real number $\epsilon > 0$. Then (4.7) guarantees that

$$|z_n| < M(\lambda_0;\phi) + \epsilon \quad \text{for } n \in \mathbf{Z}^-. \tag{4.9}$$

We claim that

$$|z_n| < M(\lambda_0;\phi) + \epsilon \quad \text{for every } n \in \mathbf{Z}. \tag{4.10}$$

Otherwise, because of (4.9), there exists an integer $n_0 > 0$ so that

$$|z_n| < M(\lambda_0;\phi) + \epsilon \quad \text{for } n \in \mathbf{Z} \text{ with } n \leq n_0 - 1$$

and

$$|z_{n_0}| \geq M(\lambda_0;\phi) + \epsilon.$$

Then, by taking into account the definition of $\mu(\lambda_0)$ and the fact that $0 < \mu(\lambda_0) < 1$, from (4.4) we obtain

$$\begin{aligned}
&M(\lambda_0;\phi) + \epsilon \\
\leq\ & |z_{n_0}| \leq \sum_{j=1}^{\infty} \lambda_0^{-j}|G_j|\,|z_{n_0-j}| + \left|1 - \frac{1}{\lambda_0}\right| \sum_{j=1}^{\infty} \lambda_0^{-j}|G_j| \left(\sum_{r=n_0-j}^{n_0-1} |z_r|\right) \\
&\qquad + \frac{1}{\lambda_0}\sum_{j=1}^{\infty} \lambda_0^{-j}|K_j| \left(\sum_{r=n_0-j}^{n_0-1} |z_r|\right) \\
\leq\ & \left[\sum_{j=1}^{\infty} \lambda_0^{-j}\left(1 + \left|1 - \frac{1}{\lambda_0}\right| j\right)|G_j| + \frac{1}{\lambda_0}\sum_{j=1}^{\infty} \lambda_0^{-j} j\,|K_j|\right][M(\lambda_0;\phi) + \epsilon] \\
\equiv\ & \mu(\lambda_0)\,[M(\lambda_0;\phi) + \epsilon] < M(\lambda_0;\phi) + \epsilon.
\end{aligned}$$

This is a contradiction and consequently our claim is true, i.e., (4.10) holds true. Since (4.10) is fulfilled for all numbers $\epsilon > 0$, we conclude that (4.8) is always satisfied. Finally, using (4.8) and taking again into account the definition of $\mu(\lambda_0)$, from (4.4) we derive, for every $n \in \mathbf{N}$,

$$\begin{aligned}
|z_n| &\leq \sum_{j=1}^{\infty} \lambda_0^{-j} |G_j| |z_{n-j}| + \left|1 - \frac{1}{\lambda_0}\right| \sum_{j=1}^{\infty} \lambda_0^{-j} |G_j| \left(\sum_{r=n-j}^{n-1} |z_r| \right) \\
&\qquad + \frac{1}{\lambda_0} \sum_{j=1}^{\infty} \lambda_0^{-j} |K_j| \left(\sum_{r=n-j}^{n-1} |z_r| \right) \\
&\leq \left[\sum_{j=1}^{\infty} \lambda_0^{-j} \left(1 + \left|1 - \frac{1}{\lambda_0}\right| j \right) |G_j| + \frac{1}{\lambda_0} \sum_{j=1}^{\infty} \lambda_0^{-j} j |K_j| \right] M(\lambda_0; \phi) \\
&\equiv \mu(\lambda_0) M(\lambda_0; \phi).
\end{aligned}$$

Consequently, (4.6) has been proved.

The proof of our theorem is complete.

Proof of Theorem 3. Consider an arbitrary initial sequence $\phi = (\phi_n)_{n \in \mathbf{Z}^-}$ in $S(\lambda_0)$ and let $(x_n)_{n \in \mathbf{Z}}$ be the solution of (E)$-$(C). Then, by Theorem 1, it holds

$$\left| \lambda_0^{-n} x_n - \frac{L(\lambda_0; \phi)}{1 + \gamma(\lambda_0)} \right| \leq \mu(\lambda_0) M(\lambda_0; \phi) \quad \text{for all } n \in \mathbf{N},$$

which leads to

$$\lambda_0^{-n} |x_n| \leq \frac{|L(\lambda_0; \phi)|}{1 + \gamma(\lambda_0)} + \mu(\lambda_0) M(\lambda_0; \phi) \quad \text{for every } n \in \mathbf{N}.$$

On the other hand, the definitions of $M(\lambda_0; \phi)$ and $N(\lambda_0; \phi)$ give

$$M(\lambda_0; \phi) \leq N(\lambda_0; \phi) + \frac{|L(\lambda_0; \phi)|}{1 + \gamma(\lambda_0)}.$$

Thus, we have

$$\lambda_0^{-n} |x_n| \leq \frac{1 + \mu(\lambda_0)}{1 + \gamma(\lambda_0)} |L(\lambda_0; \phi)| + \mu(\lambda_0) N(\lambda_0; \phi) \quad \text{for } n \in \mathbf{N}. \tag{4.11}$$

But, from the definition of $L(\lambda_0; \phi)$ it follows that

$$\begin{aligned}
|L(\lambda_0; \phi)| &\leq |\phi_0| + \sum_{j=1}^{\infty} |G_j| \left[|\phi_{-j}| + \left|1 - \frac{1}{\lambda_0}\right| \lambda_0^{-j} \left(\sum_{r=-j}^{-1} \lambda_0^{-r} |\phi_r| \right) \right] \\
&\qquad + \frac{1}{\lambda_0} \sum_{j=1}^{\infty} \lambda_0^{-j} |K_j| \left(\sum_{r=-j}^{-1} \lambda_0^{-r} |\phi_r| \right) \\
&= |\phi_0| + \sum_{j=1}^{\infty} \lambda_0^{-j} \left[\lambda_0^{-(-j)} |\phi_{-j}| + \left|1 - \frac{1}{\lambda_0}\right| \left(\sum_{r=-j}^{-1} \lambda_0^{-r} |\phi_r| \right) \right] |G_j| \\
&\qquad + \frac{1}{\lambda_0} \sum_{j=1}^{\infty} \lambda_0^{-j} \left(\sum_{r=-j}^{-1} \lambda_0^{-r} |\phi_r| \right) |K_j|,
\end{aligned}$$

which, because of the definitions of $N(\lambda_0;\phi)$ and $\mu(\lambda_0)$, yields

$$\begin{aligned} &|L(\lambda_0;\phi)| \\ \leq\ & \left[1+\sum_{j=1}^{\infty}\lambda_0^{-j}\left(1+\left|1-\frac{1}{\lambda_0}\right|j\right)|G_j|+\frac{1}{\lambda_0}\sum_{j=1}^{\infty}\lambda_0^{-j}j\,|K_j|\right]N(\lambda_0;\phi) \\ =\ & [1+\mu(\lambda_0)]\,N(\lambda_0;\phi). \end{aligned}$$

This together with (4.11) give

$$\lambda_0^{-n}|x_n| \leq \left\{\frac{[1+\mu(\lambda_0)]^2}{1+\gamma(\lambda_0)}+\mu(\lambda_0)\right\}N(\lambda_0;\phi) \quad \text{for } n\in\mathbf{N}$$

and hence, by taking into account the definition of $\Theta(\lambda_0)$, we have

$$|x_n| \leq \Theta(\lambda_0)N(\lambda_0;\phi)\lambda_0^n \quad \text{for all } n\in\mathbf{N}. \tag{4.12}$$

We have thus proved the first part of the theorem.

Next, we will establish the stability criterion contained in our theorem. Assume that $\lambda_0 \leq 1$. Consider an arbitrary bounded initial sequence $\phi = (\phi_n)_{n\in\mathbf{Z}^-}$ in S and define

$$\|\phi\| = \sup_{n\in\mathbf{Z}^-}|\phi_n|.$$

As $\lambda_0 \leq 1$, we immediately see that $\phi = (\phi_n)_{n\in\mathbf{Z}^-}$ belongs to $S(\lambda_0)$ and, in addition, that

$$N(\lambda_0;\phi) \leq \|\phi\|. \tag{4.13}$$

The solution $(x_n)_{n\in\mathbf{Z}}$ of (E)$-$(C) satisfies (4.12). By combining (4.12) and (4.13), we obtain

$$|x_n| \leq \Theta(\lambda_0)\,\|\phi\|\,\lambda_0^n \quad \text{for every } n\in\mathbf{N}. \tag{4.14}$$

Since $\lambda_0 \leq 1$, it follows from (4.14) that

$$|x_n| \leq \Theta(\lambda_0)\,\|\phi\| \quad \text{for any } n\in\mathbf{N}.$$

Thus, as $\Theta(\lambda_0) > 1$, we always have

$$|x_n| \leq \Theta(\lambda_0)\,\|\phi\| \quad \text{for all } n\in\mathbf{Z}. \tag{4.15}$$

We have proved that, for any bounded initial sequence $\phi = (\phi_n)_{n\in\mathbf{Z}^-}$ in S, the solution $(x_n)_{n\in\mathbf{Z}}$ of (E)$-$(C) satisfies (4.14) and (4.15). From (4.15) it follows that the trivial solution of (E) is stable (at 0), provided that $\lambda_0 \leq 1$. Finally, if $\lambda_0 < 1$, then (4.14) ensures that

$$\lim_{n\to\infty} x_n = 0$$

and hence the trivial solution of (E) is asymptotically stable (at 0). Finally, if $\lambda_0 < 1$, then it follows from (4.14) that the trivial solution of (E) is also exponentially stable (at 0).

The proof of the theorem has been finished.

Proof of Lemma 1. Assumption (H_1) guarantees that

$$\sum_{j=1}^{\infty}\lambda^{-j}|G_j| < \infty \quad \text{and} \quad \sum_{j=1}^{\infty}\lambda^{-j}|K_j| < \infty, \quad \text{for all } \lambda \geq \gamma$$

and hence the formula

$$F(\lambda) = (\lambda - 1)\left(1 + \sum_{j=1}^{\infty} \lambda^{-j} G_j\right) - a - \sum_{j=1}^{\infty} \lambda^{-j} K_j \quad \text{for } \lambda \geq \gamma$$

defines a continuous real-valued function on the interval $[\gamma, \infty)$. From condition (H_2) it follows that

$$F(\gamma) < 0. \tag{4.16}$$

Furthermore, for each $\lambda \geq \gamma$, we obtain

$$\begin{aligned} \left|\sum_{j=1}^{\infty} \lambda^{-j} G_j\right| &\leq \sum_{j=1}^{\infty} \lambda^{-j} |G_j| = \frac{1}{\lambda} \sum_{j=1}^{\infty} \lambda^{-j+1} |G_j| \\ &\leq \frac{1}{\lambda} \sum_{j=1}^{\infty} \gamma^{-j+1} |G_j| = \frac{\gamma}{\lambda} \sum_{j=1}^{\infty} \gamma^{-j} |G_j| \end{aligned}$$

and consequenly, by the first assumption of (H_1), we have

$$\lim_{\lambda \to \infty} \sum_{j=1}^{\infty} \lambda^{-j} G_j = 0.$$

In a similar way, one can see that

$$\lim_{\lambda \to \infty} \sum_{j=1}^{\infty} \lambda^{-j} K_j = 0.$$

So, we immediately verify that

$$F(\infty) = \infty. \tag{4.17}$$

Now, by using the hypothesis that $(K_n)_{n \in \mathbf{N} - \{0\}}$ is not eventually identically zero as well as condition (H_3), we derive for $\lambda > \gamma$

$$\begin{aligned} F'(\lambda) &= 1 + \sum_{j=1}^{\infty} \lambda^{-j} \left[1 - \left(1 - \frac{1}{\lambda}\right) j\right] G_j + \frac{1}{\lambda} \sum_{j=1}^{\infty} \lambda^{-j} j K_j \\ &\geq 1 - \sum_{j=1}^{\infty} \lambda^{-j} \left[1 + \left(1 + \frac{1}{\lambda}\right) j\right] |G_j| - \frac{1}{\lambda} \sum_{j=1}^{\infty} \lambda^{-j} j |K_j| \\ &> 1 - \sum_{j=1}^{\infty} \gamma^{-j} \left[1 + \left(1 + \frac{1}{\gamma}\right) j\right] |G_j| - \frac{1}{\gamma} \sum_{j=1}^{\infty} \gamma^{-j} j |K_j| \\ &\geq 0, \end{aligned}$$

which means that F is strictly increasing on (γ, ∞). This fact together with (4.16) and (4.17) guarantee that, in the interval (γ, ∞), the equation $F(\lambda) = 0$ (i.e., the characteristic equation $(*)$) has a unique root λ_0. Finally, by using again the hypothesis that $(K_n)_{n \in \mathbf{N} - \{0\}}$ is not eventually identically zero as well as condition

(H_3), we get

$$\begin{aligned}
& \sum_{j=1}^{\infty} \lambda_0^{-j} \left(1 + \left|1 - \frac{1}{\lambda_0}\right| j\right) |G_j| + \frac{1}{\lambda_0} \sum_{j=1}^{\infty} \lambda_0^{-j} j\, |K_j| \\
\leq\ & \sum_{j=1}^{\infty} \lambda_0^{-j} \left[1 + \left(1 + \frac{1}{\lambda_0}\right) j\right] |G_j| + \frac{1}{\lambda_0} \sum_{j=1}^{\infty} \lambda_0^{-j} j\, |K_j| \\
<\ & \sum_{j=1}^{\infty} \gamma^{-j} \left[1 + \left(1 + \frac{1}{\gamma}\right) j\right] |G_j| + \frac{1}{\gamma} \sum_{j=1}^{\infty} \gamma^{-j} j\, |K_j| \\
\leq\ & 1.
\end{aligned}$$

So, the root λ_0 of the characteristic equation (*) has the property ($P(\lambda_0)$). This completes the proof of the lemma.

Proof of Theorem 5. Let $\phi = (\phi_n)_{n\in\mathbf{Z}^-}$ be an arbitrary initial sequence in $S(\lambda_0)$, and $(x_n)_{n\in\mathbf{Z}}$ be the solution of (E)$-$(C). Define $(y_n)_{n\in\mathbf{Z}}$ and $(z_n)_{n\in\mathbf{Z}}$ as in the proof of Theorem 1. As it has been shown in the proof of Theorem 1, the fact that $(x_n)_{n\in\mathbf{Z}}$ satisfies (E) for $n \in \mathbf{N}$ is equivalent to the fact that $(z_n)_{n\in\mathbf{Z}}$ satisfies (4.4), while the initial condition (C) becomes (4.5). Furthermore, set

$$w_n = \left(\frac{\lambda_0}{\lambda_1}\right)^n z_n \quad \text{for } n \in \mathbf{Z}.$$

Then it is easy to see that (4.4) can equivalently be written as follows

$$\begin{aligned}
(4.18) \quad w_n + \sum_{j=1}^{\infty} \lambda_1^{-j} G_j w_{n-j} &= \left(1 - \frac{1}{\lambda_0}\right) \sum_{j=1}^{\infty} \lambda_0^{-j} G_j \left[\sum_{r=n-j}^{n-1} \left(\frac{\lambda_0}{\lambda_1}\right)^{n-r} w_r\right] \\
&\quad - \frac{1}{\lambda_0} \sum_{j=1}^{\infty} \lambda_0^{-j} K_j \left[\sum_{r=n-j}^{n-1} \left(\frac{\lambda_0}{\lambda_1}\right)^{n-r} w_r\right] \quad \text{for } n \in \mathbf{N}.
\end{aligned}$$

Moreover, the initial condition (4.5) is written in the following equivalent form

$$(4.19) \qquad w_n = \lambda_1^{-n} \left[\phi_n - \frac{L(\lambda_0;\phi)}{1+\gamma(\lambda_0)} \lambda_0^n\right] \quad \text{for } n \in \mathbf{Z}^-.$$

In view of the definitions of $(y_n)_{n\in\mathbf{Z}}$, $(z_n)_{n\in\mathbf{Z}}$ and $(w_n)_{n\in\mathbf{Z}}$, we have

$$(4.20) \qquad w_n = \lambda_1^{-n} \left[x_n - \frac{L(\lambda_0;\phi)}{1+\gamma(\lambda_0)} \lambda_0^n\right] \quad \text{for } n \in \mathbf{Z}.$$

From (4.19) and the definitions of $U(\lambda_0, \lambda_1; \phi)$ and $V(\lambda_0, \lambda_1; \phi)$ it follows that

$$U(\lambda_0, \lambda_1; \phi) = \inf_{s\in\mathbf{Z}^-} w_s \quad \text{and} \quad V(\lambda_0, \lambda_1; \phi) = \sup_{s\in\mathbf{Z}^-} w_s.$$

So, by taking into account (4.20), we immediately conclude that all we have to prove is that $(w_n)_{n\in\mathbf{Z}}$ satisfies

$$\inf_{s\in\mathbf{Z}^-} w_s \leq w_n \leq \sup_{s\in\mathbf{Z}^-} w_s \quad \text{for all } n \in \mathbf{N}.$$

We restrict ourselves to show that

$$(4.21) \qquad w_n \geq \inf_{s\in\mathbf{Z}^-} w_s \quad \text{for every } n \in \mathbf{N}.$$

In a similar manner, one can prove that

$$w_n \le \sup_{s\in\mathbf{Z}^-} w_s \quad \text{for every } n \in \mathbf{N}.$$

In the rest of the proof we will establish (4.21). To this end, it suffices to show that, for any real number D with $D < \inf_{s\in\mathbf{Z}^-} w_s$, it holds

$$w_n > D \quad \text{for all } n \in \mathbf{N}. \tag{4.22}$$

Let us consider an arbitrary real number D with $D < \inf_{s\in\mathbf{Z}^-} w_s$. Then we obviously have

$$w_n > D \quad \text{for } n \in \mathbf{Z}^-. \tag{4.23}$$

Assume, for the sake of contradiction, that (4.22) fails. Then, because of (4.23), there exists an integer $n_0 > 0$ so that

$$w_n > D \quad \text{for } n \in \mathbf{Z} \text{ with } n \le n_0 - 1$$

and

$$w_{n_0} \le D.$$

Hence, by using the hypothesis that $(G_n)_{n\in\mathbf{N}-\{0\}}$ and $(K_n)_{n\in\mathbf{N}-\{0\}}$ are nonpositive and that $(K_n)_{n\in\mathbf{N}-\{0\}}$ is not eventually identically zero and taking into account the assumption that $\lambda_0 \le 1$, from (4.18) we obtain

$$\begin{aligned}
D \;\ge\; w_{n_0} &= -\sum_{j=1}^{\infty}\lambda_1^{-j}G_j w_{n_0-j} + \left(1-\frac{1}{\lambda_0}\right)\sum_{j=1}^{\infty}\lambda_0^{-j}G_j\left[\sum_{r=n_0-j}^{n_0-1}\left(\frac{\lambda_0}{\lambda_1}\right)^{n_0-r} w_r\right] \\
&\quad -\frac{1}{\lambda_0}\sum_{j=1}^{\infty}\lambda_0^{-j}K_j\left[\sum_{r=n_0-j}^{n_0-1}\left(\frac{\lambda_0}{\lambda_1}\right)^{n_0-r} w_r\right] \\
&> D\left\{-\sum_{j=1}^{\infty}\lambda_1^{-j}G_j + \left(1-\frac{1}{\lambda_0}\right)\sum_{j=1}^{\infty}\lambda_0^{-j}G_j\left[\sum_{r=n_0-j}^{n_0-1}\left(\frac{\lambda_0}{\lambda_1}\right)^{n_0-r}\right]\right. \\
&\quad \left.-\frac{1}{\lambda_0}\sum_{j=1}^{\infty}\lambda_0^{-j}K_j\left[\sum_{r=n_0-j}^{n_0-1}\left(\frac{\lambda_0}{\lambda_1}\right)^{n_0-r}\right]\right\} \\
&= D\left\{-\sum_{j=1}^{\infty}\lambda_1^{-j}G_j + \left(1-\frac{1}{\lambda_0}\right)\sum_{j=1}^{\infty}\lambda_0^{-j}G_j\left[\sum_{\nu=1}^{j}\left(\frac{\lambda_0}{\lambda_1}\right)^{\nu}\right]\right. \\
&\quad \left.-\frac{1}{\lambda_0}\sum_{j=1}^{\infty}\lambda_0^{-j}K_j\left[\sum_{\nu=1}^{j}\left(\frac{\lambda_0}{\lambda_1}\right)^{\nu}\right]\right\} \\
&= D\left\{-\sum_{j=1}^{\infty}\lambda_1^{-j}G_j + \left(1-\frac{1}{\lambda_0}\right)\sum_{j=1}^{\infty}\lambda_0^{-j}G_j\frac{\frac{\lambda_0}{\lambda_1}\left[\left(\frac{\lambda_0}{\lambda_1}\right)^{j}-1\right]}{\frac{\lambda_0}{\lambda_1}-1}\right.
\end{aligned}$$

$$-\frac{1}{\lambda_0}\sum_{j=1}^{\infty}\lambda_0^{-j}K_j\frac{\frac{\lambda_0}{\lambda_1}\left[\left(\frac{\lambda_0}{\lambda_1}\right)^j-1\right]}{\frac{\lambda_0}{\lambda_1}-1}\Bigg\}$$

$$\begin{aligned}
&= \frac{D}{\lambda_0-\lambda_1}\Bigg\{-(\lambda_0-\lambda_1)\sum_{j=1}^{\infty}\lambda_1^{-j}G_j+(\lambda_0-1)\sum_{j=1}^{\infty}\lambda_0^{-j}G_j\left[\left(\frac{\lambda_0}{\lambda_1}\right)^j-1\right] \\
&\qquad -\sum_{j=1}^{\infty}\lambda_0^{-j}K_j\left[\left(\frac{\lambda_0}{\lambda_1}\right)^j-1\right]\Bigg\} \\
&= \frac{D}{\lambda_0-\lambda_1}\Bigg\{-[(\lambda_0-1)-(\lambda_1-1)]\sum_{j=1}^{\infty}\lambda_1^{-j}G_j+(\lambda_0-1)\sum_{j=1}^{\infty}\left(\lambda_1^{-j}-\lambda_0^{-j}\right)G_j \\
&\qquad -\sum_{j=1}^{\infty}\left(\lambda_1^{-j}-\lambda_0^{-j}\right)K_j\Bigg\} \\
&= \frac{D}{\lambda_0-\lambda_1}\Bigg\{\left[-(\lambda_0-1)\sum_{j=1}^{\infty}\lambda_0^{-j}G_j+\sum_{j=1}^{\infty}\lambda_0^{-j}K_j\right] \\
&\qquad -\left[-(\lambda_1-1)\sum_{j=1}^{\infty}\lambda_1^{-j}G_j+\sum_{j=1}^{\infty}\lambda_1^{-j}K_j\right]\Bigg\} \\
&= \frac{D}{\lambda_0-\lambda_1}[(\lambda_0-1-a)-(\lambda_1-1-a)] \\
&= D.
\end{aligned}$$

This contradiction shows that (4.22) holds true.

The proof of the theorem is now complete.

Proof of Theorem 6. First, let us notice that the main difference between the neutral case and the non-neutral one is the existence (in the neutral case) of the terms

$$\sum_{j=1}^{\infty}\lambda_1^{-j}G_jw_{n-j}$$

and

$$\left(1-\frac{1}{\lambda_0}\right)\sum_{j=1}^{\infty}\lambda_0^{-j}G_j\left[\sum_{r=n-j}^{n-1}\left(\frac{\lambda_0}{\lambda_1}\right)^{n-r}w_r\right]$$

in (4.18), which do not appear in the non-neutral case. In the special case of the (non-neutral) Volterra difference equation (E_0), (4.18) becomes

$$w_n=-\frac{1}{\lambda_0}\sum_{j=1}^{\infty}\lambda_0^{-j}K_j\left[\sum_{r=n-j}^{n-1}\left(\frac{\lambda_0}{\lambda_1}\right)^{n-r}w_r\right] \quad \text{for } n\in\mathbf{N}.$$

The need for assuming, in Theorem 5, that the root λ_0 of the characteristic equation (*) is such that $\lambda_0 \leq 1$ is due only to the existence of the second of the above terms in (4.18). After the above observations, we omit the proof of the theorem.

Proof of Proposition 1. Assume that there exists another positive root λ_1 of the characteristic equation (*) with $\lambda_1 < \lambda_0$ such that $(\mathrm{Q}(\lambda_1))$ holds. Clearly,

$$\sum_{j=1}^{\infty} \lambda_1^{-j} G_j \quad \text{and} \quad \sum_{j=1}^{\infty} \lambda_1^{-j} K_j \quad \text{exist in } \mathbf{R}.$$

So, since $(G_n)_{n\in\mathbf{N}-\{0\}}$ and $(K_n)_{n\in\mathbf{N}-\{0\}}$ are nonpositive, we must have

$$\sum_{j=1}^{\infty} \lambda_1^{-j} |G_j| < \infty \quad \text{and} \quad \sum_{j=1}^{\infty} \lambda_1^{-j} |K_j| < \infty.$$

(This fact can also be obtained from $(\mathrm{Q}(\lambda_1))$.) This guarantees that

$$\sum_{j=1}^{\infty} \lambda^{-j} |G_j| < \infty \quad \text{and} \quad \sum_{j=1}^{\infty} \lambda^{-j} |K_j| < \infty, \quad \text{for all } \lambda \geq \lambda_1$$

and consequently the formula

$$F(\lambda) = (\lambda - 1)\left(1 + \sum_{j=1}^{\infty} \lambda^{-j} G_j\right) - a - \sum_{j=1}^{\infty} \lambda^{-j} K_j \quad \text{for } \lambda \geq \lambda_1$$

defines a real-valued function F on the interval $[\lambda_1, \infty)$. It follows from assumption $(\mathrm{Q}(\lambda_1))$ that

$$\sum_{j=1}^{\infty} \lambda^{-j} j\, |G_j| < \infty \quad \text{and} \quad \sum_{j=1}^{\infty} \lambda^{-j} j\, |K_j| < \infty, \quad \text{for all } \lambda \geq \lambda_1,$$

which ensures that F is differentiable on $[\lambda_1, \infty)$ with

$$F'(\lambda) = 1 + \sum_{j=1}^{\infty} \lambda^{-j}\left[1 - \left(1 - \frac{1}{\lambda}\right) j\right] G_j + \frac{1}{\lambda}\sum_{j=1}^{\infty} \lambda^{-j} j K_j \quad \text{for } \lambda \geq \lambda_1.$$

Furthermore, by using the hypothesis that $(G_n)_{n\in\mathbf{N}-\{0\}}$ and $(K_n)_{n\in\mathbf{N}-\{0\}}$ are nonpositive and $(K_n)_{n\in\mathbf{N}-\{0\}}$ is not eventually identically zero, it is not difficult to check that F' is strictly increasing on the interval $[\lambda_1, 1]$. (We notice that $0 < \lambda_1 < \lambda_0 \leq 1$.)

Now, observe that $F(\lambda_1) = F(\lambda_0) = 0$, and so an application of Rolle's theorem ensures the existence of a real number ξ with $\lambda_1 < \xi < \lambda_0$ so that $F'(\xi) = 0$. Since F' is strictly increasing on $[\xi, 1]$, it follows that F' is always positive on $(\xi, 1]$. Hence, as $\xi < \lambda_0 \leq 1$, we conclude, in particular, that $F'(\lambda_0) > 0$, namely that

$$1 + \sum_{j=1}^{\infty} \lambda_0^{-j}\left[1 - \left(1 - \frac{1}{\lambda_0}\right) j\right] G_j + \frac{1}{\lambda_0}\sum_{j=1}^{\infty} \lambda_0^{-j} j K_j > 0.$$

By taking into account the fact that $(G_n)_{n\in\mathbf{N}-\{0\}}$ and $(K_n)_{n\in\mathbf{N}-\{0\}}$ are nonpositive and that $\lambda_0 \leq 1$, we see that the last inequality can equivalently be written as follows

$$1 - \sum_{j=1}^{\infty} \lambda_0^{-j}\left(1 + \left|1 - \frac{1}{\lambda_0}\right| j\right) |G_j| - \frac{1}{\lambda_0}\sum_{j=1}^{\infty} \lambda_0^{-j} j\, |K_j| > 0,$$

which means that λ_0 has the property $(\mathrm{P}(\lambda_0))$.
The proof of the proposition is complete.

Proof of Proposition 2. Let λ_1 be a positive root of the characteristic equation $(*)_0$ satisfying $(\mathrm{Q}_0(\lambda_1))$. Then it is obvious that

$$\sum_{j=1}^{\infty} \lambda_1^{-j} K_j \quad \text{exists as a real number}$$

and consequently, as $(K_n)_{n\in\mathbf{N}-\{0\}}$ is nonpositive, we have

$$\sum_{j=1}^{\infty} \lambda_1^{-j} |K_j| < \infty.$$

(Note that this fact is also a consequence of $(\mathrm{Q}_0(\lambda_1))$.) Therefore,

$$\sum_{j=1}^{\infty} \lambda^{-j} |K_j| < \infty \quad \text{for all } \lambda \geq \lambda_1$$

and so we can define the real-valued function F_0 on the interval $[\lambda_1, \infty)$ by

$$F_0(\lambda) = \lambda - 1 - a - \sum_{j=1}^{\infty} \lambda^{-j} K_j \quad \text{for } \lambda \geq \lambda_1.$$

Assumption $(\mathrm{Q}_0(\lambda_1))$ guarantees that

$$\sum_{j=1}^{\infty} \lambda^{-j} j\, |K_j| < \infty \quad \text{for all } \lambda \geq \lambda_1$$

and hence F_0 is differentiable on $[\lambda_1, \infty)$ with

$$F_0'(\lambda) = 1 + \frac{1}{\lambda} \sum_{j=1}^{\infty} \lambda^{-j} j K_j \quad \text{for } \lambda \geq \lambda_1.$$

In view of the hypothesis that $(K_n)_{n\in\mathbf{N}-\{0\}}$ is nonpositive and not eventually identically zero, we can see that F_0' is strictly increasing on the interval $[\lambda_1, \infty)$.

As $F_0(\lambda_1) = F_0(\lambda_0) = 0$, it follows from Rolle's theorem that $F_0'(\xi) = 0$ for some ξ with $\lambda_1 < \xi < \lambda_0$. Since F_0' is strictly increasing on $[\xi, \infty)$, F_0' is positive on the interval (ξ, ∞). This gives, in particular, that $F_0'(\lambda_0) > 0$, i.e.

$$1 + \frac{1}{\lambda_0} \sum_{j=1}^{\infty} \lambda_0^{-j} j K_j > 0.$$

Finally, by taking into account the fact that $(K_n)_{n\in\mathbf{N}-\{0\}}$ is nonpositive, we immediately see that λ_0 has the property $(\mathrm{P}_0(\lambda_0))$, which completes our proof.

Proof of Lemma 3. (I). Let us consider the case where $a = 0$. Then the characteristic equation $(*)$ takes the form

$$(*)' \qquad (\lambda - 1)\left(1 + \sum_{j=1}^{\infty} \lambda^{-j} G_j\right) = \sum_{j=1}^{\infty} \lambda^{-j} K_j.$$

From the hypothesis that $(K_n)_{n\in\mathbf{N}-\{0\}}$ is nonpositive and not eventually identically zero it follows that

$$\sum_{j=1}^{\infty} K_j < 0.$$

Consequently, $\lambda = 1$ cannot be a root of $(*)'$.

(II). Assume that $(*)'$ has a positive root μ with $\mu > 1$. Then

$$(\mu - 1)\left(1 + \sum_{j=1}^{\infty} \mu^{-j} G_j\right) = \sum_{j=1}^{\infty} \mu^{-j} K_j.$$

In view of the fact that $(G_n)_{n\in\mathbf{N}-\{0\}}$ is nonpositive and because of the assumption (H_4), we get

$$1 + \sum_{j=1}^{\infty} \mu^{-j} G_j \geq 1 + \sum_{j=1}^{\infty} G_j = 1 - \sum_{j=1}^{\infty} |G_j| \geq 0.$$

Thus,

$$(\mu - 1)\left(1 + \sum_{j=1}^{\infty} \mu^{-j} G_j\right) \geq 0.$$

On the other hand, since $(K_n)_{n\in\mathbf{N}-\{0\}}$ is nonpositive and not eventually identically zero, we have

$$\sum_{j=1}^{\infty} \mu^{-j} K_j < 0.$$

We have thus arrived at a contradiction.

(III). A particular consequence of assumption (H_6) is that

$$\sum_{j=1}^{\infty} j\,|K_j| < \infty. \tag{4.24}$$

Assumption (H_5) and (4.24) imply, in particular, that

$$\sum_{j=1}^{\infty} |G_j| < \infty \quad \text{and} \quad \sum_{j=1}^{\infty} |K_j| < \infty.$$

(Note that the first of the these facts can also be obtained from (H_6).) Thus, we can immediately conclude that

$$\sum_{j=1}^{\infty} \lambda^{-j}\,|G_j| < \infty \quad \text{and} \quad \sum_{j=1}^{\infty} \lambda^{-j}\,|K_j| < \infty, \qquad \text{for all } \lambda \geq 1.$$

So, the formula

$$F(\lambda) = (\lambda - 1)\left(1 + \sum_{j=1}^{\infty} \lambda^{-j} G_j\right) - a - \sum_{j=1}^{\infty} \lambda^{-j} K_j \quad \text{for } \lambda \geq 1$$

introduces a real-valued function F on the interval $[1,\infty)$. From (H_5) and (4.24) it follows that

$$\sum_{j=1}^{\infty}\lambda^{-j}j\,|G_j|<\infty \quad\text{and}\quad \sum_{j=1}^{\infty}\lambda^{-j}j\,|K_j|<\infty, \quad \text{for all } \lambda\geq 1$$

and consequently the function F is differentiable on $[1,\infty)$ with

$$F'(\lambda)=1+\sum_{j=1}^{\infty}\lambda^{-j}\left[1-\left(1-\frac{1}{\lambda}\right)j\right]G_j+\frac{1}{\lambda}\sum_{j=1}^{\infty}\lambda^{-j}jK_j \quad \text{for } \lambda\geq 1.$$

Furthermore, by the hypothesis that $(G_n)_{n\in\mathbf{N}-\{0\}}$ and $(K_n)_{n\in\mathbf{N}-\{0\}}$ are nonpositive and $(K_n)_{n\in\mathbf{N}-\{0\}}$ is not eventually identically zero, we obtain for $\lambda>1$

$$\begin{aligned}
F'(\lambda) &= 1+\sum_{j=1}^{\infty}\lambda^{-j}G_j-\left(1-\frac{1}{\lambda}\right)\sum_{j=1}^{\infty}\lambda^{-j}jG_j+\frac{1}{\lambda}\sum_{j=1}^{\infty}\lambda^{-j}jK_j\\
&= 1-\sum_{j=1}^{\infty}\lambda^{-j}\,|G_j|+\left(1-\frac{1}{\lambda}\right)\sum_{j=1}^{\infty}\lambda^{-j}j\,|G_j|-\frac{1}{\lambda}\sum_{j=1}^{\infty}\lambda^{-j}j\,|K_j|\\
&\geq 1-\sum_{j=1}^{\infty}\lambda^{-j}\,|G_j|-\frac{1}{\lambda}\sum_{j=1}^{\infty}\lambda^{-j}j\,|K_j|\\
&> 1-\sum_{j=1}^{\infty}|G_j|-\sum_{j=1}^{\infty}j\,|K_j|.
\end{aligned}$$

Hence, by assumption (H_6), we find

$$F'(\lambda)>0 \quad \text{for every } \lambda>1.$$

This implies that F is strictly increasing on the interval $(1,\infty)$. Since $(K_n)_{n\in\mathbf{N}-\{0\}}$ is nonpositive, assumption (H_7) means that

$$F(1)\geq 0. \tag{4.25}$$

Thus, the characteristic equation (*) cannot have roots in the interval $(1,\infty)$.

(IV). Assumption (H_7) means that (4.25) is true. Furthermore, assumption (H_8) guarantees, in particular, that

$$\sum_{j=1}^{\infty}\gamma^{-j}\,|G_j|<\infty \quad\text{and}\quad \sum_{j=1}^{\infty}\gamma^{-j}\,|K_j|<\infty$$

and consequently

$$\sum_{j=1}^{\infty}\lambda^{-j}\,|G_j|<\infty \quad\text{and}\quad \sum_{j=1}^{\infty}\lambda^{-j}\,|K_j|<\infty, \quad \text{for all } \lambda\geq\gamma.$$

So, the formula

$$F(\lambda)=(\lambda-1)\left(1+\sum_{j=1}^{\infty}\lambda^{-j}G_j\right)-a-\sum_{j=1}^{\infty}\lambda^{-j}K_j \quad \text{for } \lambda\geq\gamma$$

defines a real-valued function F on the interval $[\gamma, \infty)$. From assumption (H_8) it follows that

$$\sum_{j=1}^{\infty} \lambda^{-j} j \, |G_j| < \infty \quad \text{and} \quad \sum_{j=1}^{\infty} \lambda^{-j} j \, |K_j| < \infty, \quad \text{for all } \lambda \geq \gamma,$$

which ensures that the function F is differentiable on $[\gamma, \infty)$ with

$$F'(\lambda) = 1 + \sum_{j=1}^{\infty} \lambda^{-j} \left[1 - \left(1 - \frac{1}{\lambda} \right) j \right] G_j + \frac{1}{\lambda} \sum_{j=1}^{\infty} \lambda^{-j} j K_j \quad \text{for } \lambda \geq \gamma.$$

By using the hypothesis that $(G_n)_{n \in \mathbf{N} - \{0\}}$ and $(K_n)_{n \in \mathbf{N} - \{0\}}$ are nonpositive and $(K_n)_{n \in \mathbf{N} - \{0\}}$ is not eventually identically zero, we can easily verify that F' is strictly increasing on the interval $[\gamma, 1]$. Consequently,

(4.26) $$F \text{ is strictly convex on } [\gamma, 1].$$

Furthermore, we take into account the fact that $(G_n)_{n \in \mathbf{N} - \{0\}}$ and $(K_n)_{n \in \mathbf{N} - \{0\}}$ are nonpositive to conclude that assumption (H_9) means that

(4.27) $$F(\gamma) > 0,$$

while assumption (H_{10}) means that

(4.28) $$F(a + 1 - \delta) < 0.$$

A particular consequence of (4.27) is that $\lambda = \gamma$ is not a root of (*). Similarily, (4.28) guarantees, in particular, that $\lambda = a + 1 - \delta$ is not a root of (*). Moreover, from (4.25), (4.26) and (4.28) it follows that, in the interval $(a + 1 - \delta, 1]$, (*) has a unique root. Finally, (4.26), (4.27) and (4.28) ensure that, in the interval $(\gamma, a + 1 - \delta)$, (*) has also a unique root.

The lemma has now been proved.

Proof of Lemma 4. (I) and (II). Let us assume that the characteristic equation $(*)_0$ admits a positive root μ. Then

$$\mu - 1 - a = \sum_{j=1}^{\infty} \mu^{-j} K_j.$$

Since $(K_n)_{n \in \mathbf{N} - \{0\}}$ is nonpositive and not eventually identically zero, we always have

$$\sum_{j=1}^{\infty} \mu^{-j} K_j < 0.$$

So, we must have $\mu - 1 - a < 0$, i.e. $\mu < a + 1$. This shows Part (II). Moreover, it follows that $a + 1 > 0$, namely $a > -1$, and hence Part (I) has been established.

(III). From assumption $(H_8)_0$ it follows, in particular, that

$$\sum_{j=1}^{\infty} \gamma^{-j} \, |K_j| < \infty,$$

which guarantees that

$$\sum_{j=1}^{\infty} \lambda^{-j} \, |K_j| < \infty \quad \text{for all } \lambda \geq \gamma.$$

Hence, we can define the real-valued function F_0 on $[\gamma,\infty)$ by the formula

$$F_0(\lambda) = \lambda - 1 - a - \sum_{j=1}^{\infty} \lambda^{-j} K_j \quad \text{for } \lambda \geq \gamma.$$

By $(\mathrm{H}_8)_0$, we see that

$$\sum_{j=1}^{\infty} \lambda^{-j} j\, |K_j| < \infty \quad \text{for all } \lambda \geq \gamma$$

and consequently F_0 is differentiable on $[\gamma,\infty)$ with

$$F_0'(\lambda) = 1 + \frac{1}{\lambda} \sum_{j=1}^{\infty} \lambda^{-j} j K_j \quad \text{for } \lambda \geq \gamma.$$

Furthermore, the hypothesis that $(K_n)_{n\in\mathbf{N}-\{0\}}$ is nonpositive and not eventually identically zero ensures that F_0' is strictly increasing on the interval $[\gamma,\infty)$. So,

(4.29) $$F_0 \text{ is strictly convex on } [\gamma,\infty).$$

Now, as $(K_n)_{n\in\mathbf{N}-\{0\}}$ is nonpositive, assumption $(\mathrm{H}_9)_0$ means that

(4.30) $$F_0(\gamma) > 0,$$

while assumption $(\mathrm{H}_{10})_0$ means that

(4.31) $$F_0(a+1-\delta) < 0.$$

From (4.30) it follows, in particular, that $\lambda = \gamma$ is not a root of $(*)_0$, while (4.31) ensures, in particular, that $\lambda = a+1-\delta$ is not a root of $(*)_0$. Next, by taking into account the fact that $(K_n)_{n\in\mathbf{N}-\{0\}}$ is nonpositive and not eventually identically zero, we see that

(4.32) $$F_0(a+1) > 0.$$

Because of (4.29), (4.31) and (4.32), we conclude that, in the interval $(a+1-\delta, a+1)$, $(*)_0$ has a unique root. Moreover, (4.29), (4.30) and (4.31) guarantee that, in the interval $(\gamma, a+1-\delta)$, $(*)_0$ admits also a unique root.

We have thus completed the proof of our lemma.

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